

Cowen-Douglas tuples and fiber dimensions

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Abstract. Let $T \in L(X)^n$ be a Cowen-Douglas system on a Banach space X . We use functional representations of T to associate with each T -invariant subspace $Y \subseteq X$ an integer called the fiber dimension $\text{fd}(Y)$ of Y . Among other results we prove a limit formula for the fiber dimension, show that it is invariant under suitable changes of Y and deduce a dimension formula for pairs of homogeneous invariant subspaces of graded Cowen-Douglas tuples on Hilbert spaces.

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1 Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $\mathcal{H} \subseteq \mathcal{O}(\Omega, \mathbb{C}^N)$ be a functional Hilbert space of \mathbb{C}^N -valued analytic functions on Ω . The number

$$\text{fd}(\mathcal{H}) = \max_{\lambda \in \Omega} \dim \mathcal{H}_\lambda,$$

where $\mathcal{H}_\lambda = \{f(\lambda); f \in \mathcal{H}\}$, is usually referred to as the fiber dimension of \mathcal{H} . Results going back to Cowen and Douglas [6], Curto and Salinas [7] show that each Cowen-Douglas operator tuple $T \in L(H)^n$ on a Hilbert space H is locally uniformly equivalent to the tuple $M_z = (M_{z_1}, \dots, M_{z_n}) \in L(\mathcal{H})^n$ of multiplication operators with the coordinate functions on a suitable analytic functional Hilbert space \mathcal{H} . In the present note we use corresponding model theorems for Cowen-Douglas operator tuples $T \in L(X)^n$ on Banach spaces to associate with each T -invariant subspace $Y \subseteq X$ an integer $\text{fd}(Y)$ called the fiber dimension of Y . We thus extend results proved by L. Chen, G. Cheng and X. Fang in [3] for single Cowen-Douglas operators on Hilbert spaces to the case of commuting operator systems on Banach spaces.

By definition a commuting tuple $T = (T_1, \dots, T_n) \in L(X)^n$ of bounded operators on a Banach space X is a weak dual Cowen-Douglas tuple of rank $N \in \mathbb{N}$ on Ω if

$$\dim X / \sum_{i=1}^n (\lambda_i - T_i)X = N$$

for each point $\lambda \in \Omega$. We call T a dual Cowen-Douglas tuple if in addition

$$\bigcap_{\lambda \in \Omega} \sum_{i=1}^n (\lambda_i - T_i)X = \{0\}.$$

We show that weak dual Cowen-Douglas tuples $T \in L(X)^n$ admit local representations as multiplication tuples $M_z \in L(\hat{X})^n$ on suitable functional Banach spaces \hat{X} and prove that dual Cowen-Douglas tuples can be characterized as those commuting tuples $T \in L(X)^n$ that are locally jointly similar to a multiplication tuple $M_z \in L(\hat{X})^n$ on a divisible holomorphic model space \hat{X} . We use the functional representations of weak dual Cowen-Douglas tuples $T \in L(X)^n$ to associate with each linear T -invariant subspace $Y \subseteq X$ an integer $\text{fd}(Y)$ called the fiber dimension of Y .

Based on the observation that the fiber dimension $\text{fd}(Y)$ of a closed T -invariant subspace $Y \in \text{Lat}(T)$ is closely related to the Samuel multiplicity of the quotient tuple $S = T/Y \in L(X/Y)^n$ on Ω we show that the fiber dimension of $Y \in \text{Lat}(T)$ can be calculated by a limit formula

$$\text{fd}(Y) = n! \lim_{k \rightarrow \infty} \frac{\dim(Y + M_k(T - \lambda)/M_k(T - \lambda))}{k^n} \quad (\lambda \in \Omega),$$

where $M_k(T - \lambda) = \sum_{|\alpha|=k} (T - \lambda)^\alpha X$. Furthermore, we show how to calculate the fiber dimension using the sheaf model of T on Ω . We deduce that the fiber dimension is invariant against suitable changes of Y and we show that the fiber dimension for graded dual Cowen-Douglas tuples $T \in L(H)^n$ on Hilbert spaces satisfies the dimension formula

$$\text{fd}(Y_1 \vee Y_2) + \text{fd}(Y_1 \cap Y_2) = \text{fd}(Y_1) + \text{fd}(Y_2)$$

for any pair of homogeneous invariant subspaces $Y_1, Y_2 \in \text{Lat}(T)$. The proof is based on an idea from [4] (see also [3]) where a corresponding result is proved for analytic functional Hilbert spaces given by a complete Nevanlinna-Pick kernel.

2 Fiber dimension for invariant subspaces

In the following, let $\Omega \subseteq \mathbb{C}^n$ be a domain, that is, a connected open set in \mathbb{C}^n . Let D be a finite-dimensional vector space and let $M \subseteq \mathcal{O}(\Omega, D)$ be a $\mathbb{C}[z]$ -submodule. We denote the point evaluations on M by

$$\epsilon_\lambda : M \rightarrow D, f \mapsto f(\lambda) \quad (\lambda \in \Omega).$$

For $\lambda \in \Omega$, the range of ϵ_λ is a linear subspace

$$M_\lambda = \{f(\lambda); f \in M\} \subseteq D.$$

Definition 2.1. *The number*

$$\text{fd}(M) = \max_{z \in \Omega} \dim M_z$$

is called the fiber dimension of M . A point $z_0 \in \Omega$ with $\dim M_{z_0} = \text{fd}(M)$ is called a maximal point of M .

For any $\mathbb{C}[z]$ -submodule $M \subseteq \mathcal{O}(\Omega, D)$ and any point $\lambda \in \Omega$, we have

$$\sum_{i=1}^n (\lambda_i - M_{z_i})M \subseteq \ker \epsilon_\lambda.$$

Under the condition that the codimension of $\sum_{i=1}^n (\lambda_i - M_{z_i})M$ is constant on Ω , the question whether equality holds here is closely related to corresponding properties of the fiber dimension of M .

Lemma 2.2. *Consider a $\mathbb{C}[z]$ -submodule $M \subseteq \mathcal{O}(\Omega, D)$ such that there is an integer N with*

$$\dim M / \sum_{i=1}^n (\lambda_i - M_{z_i})M \equiv N$$

for all $\lambda \in \Omega$. Then $\text{fd}(M) \leq N$. If $\text{fd}(M) < N$, then

$$\sum_{i=1}^n (\lambda_i - M_{z_i})M \subsetneq \ker \epsilon_\lambda$$

for all $\lambda \in \Omega$. If $\text{fd}(M) = N$, then there is a proper analytic set $A \subseteq \Omega$ with

$$\Omega \setminus A \subseteq \{\lambda \in \Omega; \dim M_\lambda = N\} = \{\lambda \in \Omega; \sum_{i=1}^n (\lambda_i - M_{z_i})M = \ker \epsilon_\lambda\}.$$

Proof. Since the maps

$$M / \sum_{i=1}^n (\lambda_i - M_{z_i})M \rightarrow M / \ker \epsilon_\lambda \cong \text{Im } \epsilon_\lambda, [m] \mapsto [m]$$

are surjective for $\lambda \in \Omega$, it follows that $\text{fd}(M) \leq N$ and that

$$\{\lambda \in \Omega; \dim M_\lambda = N\} = \{\lambda \in \Omega; \sum_{i=1}^n (\lambda_i - M_{z_i})M = \ker \epsilon_\lambda\}.$$

Hence, if $\text{fd}(M) < N$, then $\sum_{i=1}^n (\lambda_i - M_{z_i})M \subsetneq \ker \epsilon_\lambda$ for all $\lambda \in \Omega$. A standard argument (cf. Lemma 1.4 in [8] and its proof) shows that there is a proper analytic set $A \subseteq \Omega$ such that

$$\Omega \setminus A \subseteq \{\lambda \in \Omega; \dim M_\lambda = \text{fd}(M)\}.$$

This observation completes the proof. \square

In [3] a fiber dimension was defined for invariant subspaces of dual Cowen-Douglas operators on Hilbert spaces. In the following we extend this definition to the case of weak dual Cowen-Douglas tuples on Banach spaces (see Definition 2.3).

Let X be a Banach space and let $T = (T_1, \dots, T_n) \in L(X)^n$ be a commuting tuple of bounded operators on X . For $z \in \mathbb{C}^n$, we use the notation $z - T$ both for the commuting tuple $z - T = (z_1 - T_1, \dots, z_n - T_n)$ and for the row operator

$$z - T : X^n \rightarrow X, (x_i)_{i=1}^n \mapsto \sum_{i=1}^n (z_i - T_i)x_i.$$

With this notation, we have $\sum_{i=1}^n (z_i - T_i)X = \text{Im}(z - T)$.

Definition 2.3. Let $T \in L(X)^n$ be a commuting tuple of bounded operators on X and let $\Omega \subseteq \mathbb{C}^n$ be a fixed domain. We call T a weak dual Cowen-Douglas tuple of rank $N \in \mathbb{N}$ on Ω if

$$\dim(X / \sum_{i=1}^n (z_i - T_i)X) = N$$

for all $z \in \Omega$. If in addition the condition

$$\bigcap_{z \in \Omega} \text{Im}(z - T) = \{0\}$$

holds, then T is called a dual Cowen-Douglas tuple of rank N on Ω .

If $X = H$ is a Hilbert space, then a tuple $T \in L(H)^n$ is a dual Cowen-Douglas tuple on Ω if and only if the adjoint $T^* = (T_1^*, \dots, T_n^*)$ is a tuple of class $B_n(\Omega^*)$ on the complex conjugate domain $\Omega^* = \{\bar{z}; z \in \Omega\}$ in the sense of Curto and Salinas [7]. One can show (Theorem 4.12 in [17]) that, for a weak dual Cowen-Douglas tuple $T \in L(X)^n$ on a domain $\Omega \subseteq \mathbb{C}^n$, the identity

$$\bigcap_{z \in \Omega} \text{Im}(z - T) = \bigcap_{k=0}^{\infty} \sum_{|\alpha|=k} (\lambda - T)^\alpha X$$

holds for every point $\lambda \in \Omega$. In particular, if $T \in L(X)^n$ is a dual Cowen-Douglas tuple on Ω , then it is a dual Cowen-Douglas tuple on each smaller domain $\emptyset \neq \Omega_0 \subseteq \Omega$.

Definition 2.4. Let $\Omega \subseteq \mathbb{C}^n$ be open. A holomorphic model space of rank N over Ω is a Banach space $\hat{X} \subseteq \mathcal{O}(\Omega, D)$ such that D is an N -dimensional complex vector space and

- (i) $M_z \in L(\hat{X})^n$,
- (ii) for each $\lambda \in \Omega$, the point evaluation $\epsilon_\lambda : \hat{X} \rightarrow D, \hat{x} \mapsto \hat{x}(\lambda)$, is continuous and surjective.

A holomorphic model space \hat{X} on Ω is called *divisible* if in addition, for $\hat{x} \in \hat{X}$ and $\lambda \in \Omega$ with $\hat{x}(\lambda) = 0$, there are functions $\hat{y}_1, \dots, \hat{y}_n \in \hat{X}$ with

$$\hat{x} = \sum_{i=1}^n (\lambda_i - M_{z_i}) \hat{y}_i.$$

The multiplication tuple M_z on a divisible holomorphic model space $\hat{X} \subseteq \mathcal{O}(\Omega, D)$ is easily seen to be a dual Cowen-Douglas tuple of rank $N = \dim D$ on Ω .

In the following let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank N on a fixed domain $\Omega \subseteq \mathbb{C}^n$. We extend a notion introduced in [3] to our setting.

Definition 2.5. Let $\emptyset \neq \Omega_0 \subseteq \Omega$ be a connected open subset. A CF-representation of T on Ω_0 is a $\mathbb{C}[z]$ -module homomorphism

$$\rho : X \rightarrow \mathcal{O}(\Omega_0, D)$$

with a finite-dimensional complex vector space D such that

- (i) $\ker \rho = \bigcap_{z \in \Omega} (z - T)X^n$,
- (ii) the submodule $\hat{X} = \rho X \subseteq \mathcal{O}(\Omega_0, D)$ satisfies

$$\text{fd}(\hat{X}) = \dim \hat{X} / \sum_{i=1}^n (\lambda_i - M_{z_i}) \hat{X}$$

for all $\lambda \in \Omega_0$.

Let $\mathcal{O}(\Omega_0, D)$ be equipped with its canonical Fréchet space topology. Our first aim is to show that weak dual Cowen-Douglas tuples possess sufficiently many CF-representations that are continuous and satisfy certain additional properties.

Theorem 2.6. Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank N on Ω . Then, for each point $\lambda_0 \in \Omega$, there is a CF-representation $\rho : X \rightarrow \mathcal{O}(\Omega_0, D)$ of T on a connected open neighbourhood $\Omega_0 \subseteq \Omega$ of λ_0 such that

- (i) $\rho : X \rightarrow \mathcal{O}(\Omega_0, D)$ is continuous,
- (ii) $\hat{X} = \rho(X)$ equipped with the norm $\|\rho(X)\| = \|x + \ker \rho\|$ is a divisible holomorphic model space of rank N on Ω_0 .

Proof. Let $\lambda_0 \in \Omega$ be arbitrary. Choose a linear subspace $D \subseteq X$ such that

$$X = (\lambda_0 - T)X^n \oplus D.$$

Then $\dim D = N$. The analytically parametrized complex

$$T(z) : X^n \oplus D \rightarrow X, ((x_i)_{i=1}^n, y) \mapsto \sum_{i=1}^n (z_i - T_i)x_i + y$$

of bounded operators between Banach spaces is onto at $z = \lambda_0$. By Lemma 2.1.5 in [11] there is an open polydisc $\Omega_0 \subseteq \Omega$ such that the induced map

$$\mathcal{O}(\Omega_0, X^n \oplus D) \rightarrow \mathcal{O}(\Omega_0, X), ((g_i)_{i=1}^n, h) \mapsto \sum_{i=1}^n (z_i - T_i)g_i + h$$

is onto. In particular, for each $z \in \Omega_0$, the linear map

$$D \rightarrow X / \sum_{i=1}^n (z_i - T_i)X, x \mapsto [x]$$

is surjective between N -dimensional complex vector space. Hence these maps are isomorphisms. But then, for each $x \in X$ and $z \in \Omega_0$, there is a unique vector $x(z) \in D$ with $x - x(z) \in \sum_{i=1}^n (z_i - T_i)X$. By construction, for each $x \in X$, the mapping $\Omega_0 \rightarrow D, z \mapsto x(z)$, is analytic. The induced mapping

$$\rho : X \rightarrow \mathcal{O}(\Omega_0, D), x \mapsto x(\cdot)$$

is linear with

$$\ker \rho = \bigcap_{z \in \Omega_0} \sum_{i=1}^n (z_i - T_i)X = \bigcap_{z \in \Omega} \sum_{i=1}^n (z_i - T_i)X.$$

For $x \in X$, $z \in \Omega_0$ and $j = 1, \dots, n$,

$$T_j x - z_j x(z) = T_j(x - x(z)) - (z_j - T_j)x(z) \in \sum_{i=1}^n (z_i - T_i)X.$$

Hence ρ is a $\mathbb{C}[z]$ -module homomorphism. Equipped with the norm $\|\rho(x)\| = \|x + \ker \rho\|$, the space $\hat{X} = \rho(X)$ is a Banach space and $M_z \in L(\hat{X})^n$ is a commuting tuple of bounded operators on \hat{X} . By definition

$$\rho(x) \equiv x \quad \text{for } x \in D.$$

Hence the point evaluations $\epsilon_z : \hat{X} \rightarrow D$ ($z \in \Omega_0$) are surjective. Since the mappings

$$q_z : D \rightarrow X / \sum_{i=1}^n (z_i - T_i)X, x \mapsto [x] \quad (z \in \Omega_0)$$

are topological isomorphisms and since the compositions

$$X \rightarrow X / \sum_{i=1}^n (z_i - T_i)X, x \mapsto q_z(\epsilon_z(\rho(x))) = [x]$$

are continuous, it follows that the point evaluations $\epsilon_z : \hat{X} \rightarrow D$ ($z \in \Omega_0$) are continuous. Thus we have shown that $\hat{X} \subseteq \mathcal{O}(\Omega_0, D)$ with the norm induced by ρ is a holomorphic model space.

To see that \hat{X} is divisible, fix a vector $x \in X$ and a point $\lambda \in \Omega_0$ such that $x(\lambda) = 0$. Then there are vectors $x_1, \dots, x_n \in X$ with $x = \sum_{i=1}^n (\lambda_i - T_i)x_i$. Hence

$$\rho(x) = \sum_{i=1}^n (\lambda_i - z_i)\rho(x_i) \in \sum_{i=1}^n (\lambda_i - M_{z_i})\hat{X}.$$

To conclude the proof, it suffices to observe that

$$\dim(\hat{X} / \sum_{i=1}^n (\lambda_i - M_{z_i})\hat{X}) = \dim(\hat{X} / \ker \epsilon_\lambda) = \dim(\text{Im } \epsilon_\lambda) = \dim D = N$$

for all $z \in \Omega_0$. □

Note that, for a dual Cowen-Douglas tuple $T \in L(X)^n$ on a Banach space X , the mappings $\rho : X \rightarrow \hat{X} \subseteq \mathcal{O}(\Omega_0, D)$ constructed in the previous proof are isometric joint similarities between $T \in L(X)^n$ and the tuples $M_z \in L(\hat{X})^n$ on the divisible holomorphic model space $\hat{X} \subseteq \mathcal{O}(\Omega_0, D)$.

Corollary 2.7. *Let $T \in L(X)^n$ be a commuting tuple on a complex Banach space and let $\Omega \subseteq \mathbb{C}^n$ be a domain. The tuple T is a dual Cowen-Douglas tuple of rank N on Ω if and only if, for each point $\lambda \in \Omega$, there exist a connected open neighbourhood $\Omega_0 \subseteq \Omega$ of λ and a joint similarity between T and the multiplication tuple $M_z \in L(\hat{X})^n$ on a divisible holomorphic model space \hat{X} of rank N on Ω_0 .*

Proof. The necessity of the stated condition follows from Theorem 2.6 and the subsequent remarks. Since the tuple $M_z \in L(\hat{X})^n$ on a divisible holomorphic model space of rank N is a dual Cowen-Douglas tuple of rank N , and since the same is true for every tuple similar to $M_z \in L(\hat{X})^n$, also the sufficiency is clear. □

The preceding result should be compared with Corollary 4.39 in [17], where a characterization of dual Cowen-Douglas tuple on suitable admissible domains in \mathbb{C}^n is obtained.

There is a canonical way to associate with each weak dual Cowen-Douglas tuple of rank N on $\Omega \subseteq \mathbb{C}^n$ a dual Cowen-Douglas tuple of rank N .

Corollary 2.8. *Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^n$. Then the quotient tuple*

$$T^{CD} = T / \bigcap_{z \in \Omega} \sum_{i=1}^n (z_i - T_i)X$$

defines a dual Cowen-Douglas tuple of rank N on Ω .

Proof. Let $z_0 \in \Omega$ be arbitrary. Choose a CF-representation $\rho : X \rightarrow \mathcal{O}(\Omega_0, D)$ as in Theorem 2.6. Then $\hat{X} = \rho(X) \subseteq \mathcal{O}(\Omega_0, D)$ is a divisible holomorphic model space of rank N on Ω_0 . Since

$$\ker \rho = \bigcap_{z \in \Omega} \sum_{i=1}^n (z_i - T_i)X,$$

the mapping ρ induces a similarity between T^{CD} and $M_z \in L(\hat{X})^n$. By Corollary 2.7 the tuple T^{CD} is a dual Cowen-Douglas tuple of rank N on Ω . \square

As before, let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^n$. We denote by $\text{Lat}(T)$ the set of closed subspaces $Y \subseteq X$ which are invariant under each component T_i of T . Our next aim is to show that, for $Y \in \text{Lat}(T)$, the fiber dimension of Y can be defined as

$$\text{fd}(Y) = \text{fd}(\rho(Y)),$$

where ρ is an arbitrary CF-representation of T . We have of course to show that the number $\text{fd}(\rho(Y))$ is independent of the chosen CF-representation ρ . In the first step, we use an argument from [3] to show that $\text{fd}(\rho_1(Y)) = \text{fd}(\rho_2(Y))$ for each pair of CF-representations ρ_1, ρ_2 over domains $\Omega_1, \Omega_2 \subseteq \Omega$ with non-trivial intersection.

Lemma 2.9. *Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ be domains with $\Omega_1 \cap \Omega_2 \neq \emptyset$ and let $M_i \subseteq \mathcal{O}(\Omega_i, D_i)$ be $\mathbb{C}[z]$ -submodules with finite-dimensional complex vector spaces D_i such that*

$$\text{fd}(M_i) = \dim M_i / (\lambda - M_z)M_i^n \quad (i = 1, 2, \lambda \in \Omega_i).$$

Suppose that there is a $\mathbb{C}[z]$ -module isomorphism $U : M_1 \rightarrow M_2$. Then, for any submodule $M \subseteq M_1$, we have

$$\text{fd}(M) = \text{fd}(UM).$$

Proof. Using Lemma 1.4 in [8] and elementary properties of analytic sets, we can choose a proper analytic subset $A \subseteq \Omega_1 \cap \Omega_2$ such that each point $\lambda \in (\Omega_1 \cap \Omega_2) \setminus A$ is a maximal point for M, M_1 and UM . Fix such a point λ . If $f, g \in M$ are functions with $f(\lambda) = g(\lambda)$, then by Lemma 2.2 applied to M_1 , there are functions $h_1, \dots, h_n \in M_1$ such that $f - g = \sum_{i=1}^n (\lambda_i - M_{z_i})h_i$. But then also

$$U(f - g) = \sum_{i=1}^n (\lambda_i - M_{z_i})U h_i.$$

Hence we obtain a well-defined surjective linear map $U_\lambda : M_\lambda \rightarrow (UM)_\lambda$ by setting

$$U_\lambda x = (Uf)(\lambda) \text{ if } f \in M \text{ with } f(\lambda) = x.$$

It follows that $\text{fd}(M) = \dim M_\lambda \geq \dim(UM)_\lambda = \text{fd}(UM)$. By applying the same argument to U^{-1} and UM instead of U and M we find that also $\text{fd}(UM) \geq \text{fd}(M)$. \square

If $\rho_i : X \rightarrow \mathcal{O}(\Omega_i, D_i)$ ($i = 1, 2$) are CF-representations on domains $\Omega_i \subseteq \Omega$ with non-trivial intersection $\Omega_1 \cap \Omega_2 \neq \emptyset$, then the submodules $M_i = \rho_i X \subseteq \mathcal{O}(\Omega_i, D_i)$ are canonically isomorphic

$$M_1 \cong X / \ker \rho_1 = X / \ker \rho_2 \cong M_2$$

as $\mathbb{C}[z]$ -modules. As an application of the previous result one obtains that

$$\text{fd}(\rho_1 Y) = \text{fd}(\rho_2 Y)$$

for each linear subspace $Y \subseteq X$ which is invariant for T .

Theorem 2.10. *Let $\rho_i : X \rightarrow \mathcal{O}(\Omega_i, D_i)$ ($i = 1, 2$) be CF-representations of T on domains $\Omega_i \subseteq \Omega$. Then*

$$\text{fd}(\rho_1 Y) = \text{fd}(\rho_2 Y)$$

for each linear subspace $Y \subseteq X$ which is invariant for T .

Proof. Since Ω is connected, we can choose a continuous path $\gamma : [0, 1] \rightarrow \Omega$ such that $\gamma(0) \in \Omega_1$ and $\gamma(1) \in \Omega_2$. By Theorem 2.6 there is a family $(\rho_z)_{z \in \text{Im} \gamma}$ of CF-representations $\rho_z : X \rightarrow \mathcal{O}(\Omega_z, D_z)$ of T on connected open neighbourhoods $\Omega_z \subseteq \Omega$ of the points $z \in \text{Im} \gamma$ such that $\rho_{\gamma(0)} = \rho_1$ and $\rho_{\gamma(1)} = \rho_2$. Using the fact that there is a positive number $\delta > 0$ such that each set $A \subseteq [0, 1]$ of diameter less than δ is completely contained in one of the sets $\gamma^{-1}(\Omega_z)$ (see e.g. Lemma 3.7.2 in [15]), one can choose a sequence of points $z_1 = \gamma(0), z_2, \dots, z_n = \gamma(1)$ in $\text{Im} \gamma$ such that $\Omega_{z_i} \cap \Omega_{z_{i+1}} \neq \emptyset$ for $i = 1, \dots, n-1$. Let $Y \subseteq X$ be a linear T -invariant subspace. By the remarks following Lemma 2.9 we obtain that

$$\text{fd}(\rho_1 Y) = \text{fd}(\rho_{z_2} Y) = \dots = \text{fd}(\rho_2 Y)$$

as was to be shown. \square

Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^n$ and let $Y \subseteq X$ be a linear subspace that is invariant for T . In view of Theorem 2.10 we can define the fiber dimension of Y by

$$\text{fd}(Y) = \text{fd}(\rho Y),$$

where $\rho : X \rightarrow \mathcal{O}(\Omega_0, D)$ is an arbitrary CF-representation of T . We shall mainly be interested in the fiber dimension of closed invariant subspaces $Y \in \text{Lat}(T)$, but the reader should observe that the definition makes perfect sense for linear T -invariant subspaces $Y \subseteq X$. Since by Theorem 2.6 there

are always continuous CF-representations $\rho : X \rightarrow \mathcal{O}(\Omega_0, D)$ and since in this case the inclusions

$$\epsilon_\lambda(\rho(\overline{Y})) \subseteq \overline{\epsilon_\lambda(\rho(Y))} = \epsilon_\lambda(\rho(Y))$$

hold for all $\lambda \in \Omega_0$, it follows that $\text{fd}(Y) = \text{fd}(\overline{Y})$ for each linear T -invariant subspace $Y \subseteq X$.

It follows from Theorem 2.6 that $\text{fd}(X) = N$. In general, the fiber dimension $\text{fd}(Y)$ of a linear T -invariant subspace $Y \subseteq X$ is an integer in $\{0, \dots, N\}$ which depends on Y in a monotone way. Obviously, $\text{fd}(Y) = 0$ if and only if

$$Y \subseteq \ker \rho = \bigcap_{z \in \Omega} (z - T)X^n.$$

We conclude this section with an alternative characterization of CF-representations.

Corollary 2.11. *Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^n$ and let $\rho : X \rightarrow \mathcal{O}(\Omega_0, D)$ be a $\mathbb{C}[z]$ -module homomorphism on a domain $\emptyset \neq \Omega_0 \subseteq \Omega$ with a finite-dimensional vector space D such that*

$$\ker \rho = \bigcap_{z \in \Omega} (z - T)X^n.$$

Then ρ is a CF-representation of T if and only if $\text{fd}(\rho X) = N$.

Proof. Suppose that $\text{fd}(\rho X) = N$. Define $\hat{X} = \rho(X)$. Since the maps

$$X/(\lambda - T)X^n \rightarrow \hat{X}/(\lambda - M_z)\hat{X}^n, [x] \mapsto [\rho x]$$

and

$$\hat{X}/(\lambda - M_z)\hat{X}^n \rightarrow \hat{X}_\lambda, [f] \mapsto f(\lambda)$$

are surjective for each $\lambda \in \Omega_0$, it follows that

$$\dim \hat{X}/(\lambda - M_z)\hat{X}^n \leq N$$

for all $\lambda \in \Omega_0$ and that equality holds on $\Omega_0 \setminus A$ with a suitable proper analytic subset $A \subseteq \Omega_0$. Equipped with the norm $\|\rho(x)\| = \|x + \ker \rho\|$, the space \hat{X} is a Banach space and $M_z \in L(\hat{X})^n$ is a commuting tuple of bounded operators on \hat{X} . A result of Kaballo (Satz 1.5 in [13]) shows that the set

$$\{\lambda \in \Omega_0; \dim \hat{X}/(\lambda - M_z)\hat{X}^n > \min_{\mu \in \Omega_0} \dim \hat{X}/(\mu - M_z)\hat{X}^n\}$$

is a proper analytic subset of Ω_0 . Combining these results we find that

$$\dim \hat{X}/(\lambda - M_z)\hat{X}^n = N$$

for all $\lambda \in \Omega_0$. Hence ρ is a CF-representation of T .

Conversely, if ρ is a CF-representation of T , then $\text{fd}(\rho X) = N$ by the remarks preceding the corollary. \square

3 A limit formula for the fiber dimension

Let $\Omega \subseteq \mathbb{C}^n$ be a domain with $0 \in \Omega$ and let D be a finite-dimensional complex vector space. For $k \in \mathbb{N}$, let us consider the mapping $T_k : \mathcal{O}(\Omega, D) \rightarrow \mathcal{O}(\Omega, D)$ which associates with each function $f \in \mathcal{O}(\Omega, D)$ its k -th Taylor polynomial, that is,

$$T_k(f)(z) = \sum_{|\alpha| \leq k} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha.$$

In [8] (Lemma 1.4) it was shown that, for a given $\mathbb{C}[z]$ -submodule, there is a proper analytic subset $A \subseteq \Omega$ such that

$$\dim M_z = \max_{w \in \Omega} \dim M_w = n! \lim_{k \rightarrow \infty} \frac{\dim T_k(M)}{k^n}$$

holds for all $z \in \Omega \setminus A$.

Based on this observation, we will deduce a similar limit formula for the fiber dimension of invariant subspaces of weak Cowen-Douglas tuples on Ω . Given a commuting tuple $T \in L(X)^n$ of bounded operators on a Banach space X , we write

$$K^\bullet(T, X) : 0 \longrightarrow \Lambda^0(X) \xrightarrow{\delta_T^0} \Lambda^1(X) \xrightarrow{\delta_T^1} \dots \xrightarrow{\delta_T^{n-1}} \Lambda^n(X) \longrightarrow 0$$

for the Koszul complex of T (cf. Section 2.2 in [11]). For $i = 0, \dots, n$, let

$$H^i(T, X) = \ker(\delta_T^i) / \text{Im}(\delta_T^{i-1})$$

be the i -th cohomology group of $K^\bullet(T, X)$. There is a canonical isomorphism $H^n(T, X) \cong X / \sum_{i=1}^n T_i X$ of complex vector spaces.

In the following, given a commuting operator tuple $T \in L(X)^n$ and an invariant subspace $Y \in \text{Lat}(T)$, we denote by

$$R = T|_Y \in L(Y)^n, S = T/Y \in L(Z)^n$$

the restriction of T to Y and the quotient of T modulo Y on $Z = X/Y$. The inclusion $i : X \rightarrow Y$ and the quotient map $q : X \rightarrow Z$ induce a short exact sequence of complexes

$$0 \longrightarrow K^\bullet(z - R, Y) \xrightarrow{i} K^\bullet(z - T, X) \xrightarrow{q} K^\bullet(z - S, Z) \longrightarrow 0.$$

It is a standard fact from homological algebra that there are connecting homomorphisms $d_z^i : H^i(z - S, Z) \rightarrow H^{i+1}(z - R, Y)$ ($i = 0, \dots, n-1$) such that the induced sequence of cohomology spaces

$$\begin{aligned}
0 &\longrightarrow H^0(z - R, Y) \xrightarrow{i} H^0(z - T, X) \xrightarrow{q} H^0(z - S, Z) \\
&\xrightarrow{d_z^0} H^1(z - R, Y) \xrightarrow{i} H^1(z - T, X) \xrightarrow{q} H^1(z - S, Z) \\
&\xrightarrow{d_z^1} H^2(z - R, Y) \longrightarrow \dots \\
&\xrightarrow{d_z^{n-1}} H^n(z - R, Y) \xrightarrow{i} H^n(z - T, X) \xrightarrow{q} H^n(z - S, Z) \longrightarrow 0
\end{aligned}$$

is exact again. In particular, we obtain

$$\begin{aligned}
\text{Im}(d_z^{n-1}) &= \ker(H^n(z - R, Y) \xrightarrow{i} H^n(z - T, X)) \\
&= (Y \cap (z - T)X^n) / (z - R)Y^n.
\end{aligned}$$

Lemma 3.1. *Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^n$ and let $Y \in \text{Lat}(T)$ be a closed invariant subspace of T . Then there is a proper analytic subset $A \subseteq \Omega$ such that*

$$\dim H^n(\lambda - S, Z) = N - \text{fd}(Y)$$

for all $\lambda \in \Omega \setminus A$.

Proof. Choose a CF-representation $\rho : X \rightarrow \mathcal{O}(\Omega_0, D)$ of T on some domain $\Omega_0 \subseteq \Omega$ as in Theorem 2.6. Let $Y \in \text{Lat}(T)$ be arbitrary. Define $\hat{X} = \rho(X)$ and $\hat{Y} = \rho(Y)$. Since the compositions

$$Y^n \xrightarrow{\lambda - R} Y \xrightarrow{\rho} \mathcal{O}(\Omega_0, D) \xrightarrow{\epsilon_\lambda} D \quad (\lambda \in D)$$

are zero, we obtain well-defined surjective linear maps

$$\delta_\lambda : H^n(\lambda - R, Y) \rightarrow \hat{Y}_\lambda, [y] \mapsto \rho(y)(\lambda).$$

Obviously, for each $\lambda \in \Omega$, the inclusion

$$\text{Im} d_\lambda^{n-1} = (Y \cap (\lambda - T)X^n) / (\lambda - R)Y^n \subseteq \ker \delta_\lambda$$

holds. To see that also the reverse inclusion holds, fix an element $y \in Y$ with $\rho(y)(\lambda) = 0$. Since \hat{X} is a divisible holomorphic model space, there are vectors $x_1, \dots, x_n \in X$ with

$$\rho(y) = \sum_{i=1}^n (\lambda_i - M_{z_i}) \rho(x_i) = \rho\left(\sum_{i=1}^n (\lambda_i - T_i) x_i\right).$$

But then

$$y - \sum_{i=1}^n (\lambda_i - T_i)x_i \in \bigcap_{z \in \Omega} (z - T)X^n$$

and hence $y \in Y \cap (\lambda - T)X^n$. Thus, for each $\lambda \in \Omega$, we obtain an exact sequence

$$H^{n-1}(\lambda - S, Z) \xrightarrow{d_\lambda^{n-1}} H^n(\lambda - R, Y) \xrightarrow{\delta_\lambda} \hat{Y}_\lambda \rightarrow 0.$$

Using the exactness of these sequences and of the long exact cohomology sequences explained in the section leading to Lemma 2.1, we find that

$$\begin{aligned} & \dim H^n(\lambda - S, Z) \\ &= \dim H^n(\lambda - T, X) - \dim H^n(\lambda - R, Y)/d_\lambda^{n-1} H^{n-1}(\lambda - S, Z) \\ &= N - \dim \hat{Y}_\lambda \end{aligned}$$

for all $\lambda \in \Omega$. Hence the assertion follows. \square

By the cited result of Khablo (Satz 1.5 in [13]), in the setting of Lemma 2.1, the set

$$\{\lambda \in \Omega; \dim H^n(\lambda - S, Z) > \min_{\mu \in \Omega} \dim H^n(\mu - S, Z)\}$$

is an analytic subset of Ω . It is well known that the minimum occurring here can be interpreted as a suitable Samuel multiplicity of the tuples $S - \mu$ for $\mu \in \Omega$. Let us recall the necessary details.

For simplicity, we only consider the case where Ω is a domain in \mathbb{C}^n with $0 \in \Omega$. For an arbitrary tuple $T \in L(X)^n$ of bounded operators on a Banach space X with

$$\dim H^n(T, X) < \infty,$$

all the spaces $M_k(T) = \sum_{|\alpha|=k} T^\alpha X$ ($k \in \mathbb{N}$) are finite codimensional in X and the limit

$$c(T) = n! \lim_{k \rightarrow \infty} \frac{\dim X/M_k(T)}{k^n}$$

exists. This number is referred to as the Samuel multiplicity of T . For each domain $\Omega \subseteq \mathbb{C}^n$ with $0 \in \Omega$ and $\dim H^n(\lambda - T, X) < \infty$ for all $\lambda \in \Omega$, there is a proper analytic subset $A \subseteq \Omega$ such that

$$c(T) = \dim H^n(\lambda - T, X) < \dim H^n(\mu - T, X)$$

for all $\lambda \in \Omega \setminus A$ and $\mu \in A$ (see Corollary 3.6 in [9]). In particular, if $S \in L(Z)^n$ is as in Lemma 2.1 and $0 \in \Omega$, then the formula

$$c(S) = N - \text{fd}(Y)$$

holds. Hence the following result from [8] allows us to deduce the announced limit formula for the fiber dimension.

Lemma 3.2. (Lemma 1.6 in [8]) *Let $T \in L(X)^n$ be a commuting tuple of bounded operators on a Banach space X , let $Y \in \text{Lat}(T)$ be a closed invariant subspace and let $S = T/Y \in L(Z)^n$ be the induced quotient tuple on $Z = X/Y$. Suppose that*

$$\dim H^n(T, X) < \infty.$$

Then the Samuel multiplicities of T and S satisfy the relation

$$c(S) = c(T) - n! \lim_{k \rightarrow \infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^n}.$$

As a direct application we obtain a corresponding formula for the fiber dimension.

Corollary 3.3. *Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^n$ with $0 \in \Omega$, and let $Y \in \text{Lat}(T)$ be a closed invariant subspace for T . Then the formula*

$$\text{fd}(Y) = n! \lim_{k \rightarrow \infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^n}$$

holds.

Proof. It suffices to observe that in the setting of Corollary 3.3 the identity $c(T) = N$ holds and then to compare the formula from Lemma 3.2 with the formula

$$c(S) = N - \text{fd}(Y)$$

deduced in the section leading to Lemma 3.2. □

For weak dual Cowen-Douglas tuples $T \in L(X)^n$ on general domains $\Omega \subseteq \mathbb{C}^n$ (not necessarily containing 0), the above formula for $\text{fd}(Y)$ remains true if on the right-hand side the spaces $M_k(T)$ are replaced by the spaces $M_k(T - \lambda_0)$ with $\lambda_0 \in \Omega$ arbitrary. This follows by an elementary translation argument.

If in Corollary 3.3 the space X is a Hilbert space and if we write P_k for the orthogonal projections onto the subspaces $M_k(T)^\perp$, then there are canonical vector space isomorphisms

$$(Y + M_k(T))/M_k(T) \rightarrow P_k Y, [y] \mapsto P_k Y.$$

Thus the resulting formula

$$\text{fd}(Y) = n! \lim_{k \rightarrow \infty} \frac{\dim(P_k Y)}{k^n}$$

extends Theorem 19 in [3].

In the final result of this section we show that the fiber dimension $\text{fd}(Y)$ is invariant under sufficiently small changes of the space Y . For given invariant subspaces $Y_1, Y_2 \in \text{Lat}(T)$ with $Y_1 \subseteq Y_2$, we write $\sigma(T, Y_2/Y_1)$ for the Taylor spectrum of the quotient tuple induced by T on Y_2/Y_1 .

Corollary 3.4. *Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^n$. Suppose that $Y_1, Y_2 \in \text{Lat}(T)$ are closed T -invariant subspaces with $Y_1 \subseteq Y_2$ and $\Omega \cap (\mathbb{C}^n \setminus \sigma(T, Y_2/Y_1)) \neq \emptyset$. Then $\text{fd}(Y_1) = \text{fd}(Y_2)$.*

Proof. By Lemma 2.1 there is a point $\lambda \in \Omega \cap (\mathbb{C}^n \setminus \sigma(T, Y_1/Y_2))$ with

$$\dim H^n(\lambda - T/Y_i, X/Y_i) = N - \text{fd}(Y_i)$$

for $i = 1, 2$. Using the long exact cohomology sequences induced by the canonical exact sequence

$$0 \rightarrow Y_2/Y_1 \rightarrow Y/Y_1 \rightarrow Y/Y_2 \rightarrow 0$$

one finds that the n -th cohomology spaces of $\lambda - T/Y_1$ and $\lambda - T/Y_2$ are isomorphic. Hence we obtain that $\text{fd}(Y_1) = \text{fd}(Y_2)$. \square

To make the above proof work, it suffices that there is a point in Ω which is not contained in the right spectrum of the quotient tuple induced by T on Y_2/Y_1 (cf. Section 2.6 in [11]). The hypotheses of Corollary 2.4 are satisfied for instance if $\dim(Y_2/Y_1) < \infty$. Thus Corollary 2.4 can be seen as an extension of Proposition 2.5 in [5].

4 Analytic Samuel multiplicity

We briefly indicate an alternative way to calculate fiber dimensions which extends a corresponding idea from [3]. Let $T \in L(X)^n$ be a commuting tuple of bounded operators on a Banach space X and let $\Omega \subseteq \mathbb{C}^n$ be a domain such that

$$\dim H^n(\lambda - T, X) < \infty$$

for all $\lambda \in \Omega$. For simplicity, we again assume that $0 \in \Omega$. By Corollary 2.2 in [9] the quotient sheaf

$$\mathcal{H}_T = \mathcal{O}_\Omega^X / (z - T)\mathcal{O}_\Omega^{X^n}$$

of the sheaf of all analytic X -valued functions on Ω is a coherent analytic sheaf on Ω . Let $Y \in \text{Lat}(T)$ be a closed invariant subspace for T . As before denote by $R = T|_Y \in L(Y)^n$ the restriction of T and by $S = T/Y \in L(Z)^n$ the quotient tuple induced by T on $Z = X/Y$. Let $i : Y \rightarrow X$ and $q : X \rightarrow Z$ be the inclusion and quotient map, respectively. Then

$$0 \rightarrow K^\bullet(z - R, \mathcal{O}_\Omega^Y) \xrightarrow{i} K^\bullet(z - T, \mathcal{O}_\Omega^X) \xrightarrow{q} K^\bullet(z - S, \mathcal{O}_\Omega^Z) \rightarrow 0$$

is a short exact sequence of complexes of analytic sheaves on Ω . Passing to stalks and using the induced long exact cohomology sequences, one finds

that the upper horizontal in the commutative diagram

$$\begin{array}{ccccc} \mathcal{H}_R & \xrightarrow{i} & \mathcal{H}_T & \xrightarrow{q} & \mathcal{H}_S \rightarrow 0 \\ \uparrow \pi_Y & & \uparrow \pi_X & & \\ \mathcal{O}_\Omega^Y & \xrightarrow{i} & \mathcal{O}_\Omega^X & & \end{array}$$

is an exact sequence of analytic sheaves. Here π_Y and π_X denote the canonical quotient maps. The sheaf $\mathcal{M} = \pi_X(i\mathcal{O}_\Omega^Y)$ is the kernel of the surjective sheaf homomorphism

$$\mathcal{H}_T \xrightarrow{q} \mathcal{H}_S.$$

Since \mathcal{H}_T and \mathcal{H}_S are coherent, also the sheaf \mathcal{M} is a coherent analytic sheaf on Ω (Satz 26.13 in [14]). Hence

$$0 \rightarrow \mathcal{M}_0 \xrightarrow{i} \mathcal{H}_{T,0} \xrightarrow{q} \mathcal{H}_{S,0} \rightarrow 0$$

is an exact sequence of Noetherian \mathcal{O}_0 -modules. For a Noetherian \mathcal{O}_0 -module E , let us denote by $e_{\mathcal{O}_0}(E)$ its analytic Samuel multiplicity, that is, the multiplicity of E with respect to the multiplicity system (z_1, \dots, z_n) on E (see Section 7.4 in [16]). Since the analytic Samuel multiplicity is additive with respect to short exact sequences of Noetherian \mathcal{O}_0 -modules (Theorem 7.5 in [16]), it follows that

$$e_{\mathcal{O}_0}(\mathcal{H}_{T,0}) = e_{\mathcal{O}_0}(\mathcal{M}_0) + e_{\mathcal{O}_0}(\mathcal{H}_{S,0}).$$

By Corollary 4.1 in [9] the analytic Samuel multiplicities $e_{\mathcal{O}_0}(\mathcal{H}_{T,0})$ and $e_{\mathcal{O}_0}(\mathcal{H}_{S,0})$ coincide with the Samuel multiplicities $c(T)$ and $c(S)$ as defined in Section 2. Thus we obtain the identity

$$c(T) = e_{\mathcal{O}_0}(\mathcal{M}_0) + c(S).$$

By Theorem 8.5 in [16] the analytic Samuel multiplicity $e_{\mathcal{O}_0}(\mathcal{M}_0)$ can also be calculated as the Euler characteristic $\chi(K^\bullet(z, \mathcal{M}_0))$ of the Koszul complex of the multiplication operators with z_1, \dots, z_n on \mathcal{M}_0 .

Summarizing we obtain the following result.

Theorem 4.1. *Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple on a domain $\Omega \subseteq \mathbb{C}^n$ with $0 \in \Omega$ and let $Y \in \text{Lat}(T)$ be a closed invariant subspace for T . Then with the notation from above, the fiber dimension of Y can be calculated as*

$$\text{fd}(Y) = n! \lim_{k \rightarrow \infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^n} = e_{\mathcal{O}_0}(\mathcal{M}_0).$$

5 A lattice formula for the fiber dimension

Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^n$ and let $Y_1, Y_2 \in \text{Lat}(T)$ be closed invariant subspaces. A natural problem studied in [3] is to find conditions under which the dimension formula

$$\text{fd}(Y_1) + \text{fd}(Y_2) = \text{fd}(Y_1 \vee Y_2) + \text{fd}(Y_1 \cap Y_2)$$

holds. Note that, for a dual Cowen-Douglas tuple of rank 1, the validity of this formula for all closed invariant subspaces Y_1, Y_2 is equivalent to the condition that any two non-zero closed invariant subspaces Y_1, Y_2 have a non-trivial intersection. As observed in [3] elementary linear algebra can be used to obtain at least an inequality.

Lemma 5.1. *Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple on a domain $\Omega \subseteq \mathbb{C}^n$ and let $Y_1, Y_2 \subseteq X$ be linear T -invariant subspaces. Then the inequality*

$$\text{fd}(Y_1) + \text{fd}(Y_2) \geq \text{fd}(Y_1 + Y_2) + \text{fd}(Y_1 \cap Y_2)$$

holds.

Proof. Let $\rho : X \rightarrow \mathcal{O}(\Omega_0, D)$ be a CF-representation of T on a domain $\Omega_0 \subseteq \Omega$. It suffices to observe that, for each point $\lambda \in \Omega_0$, the estimate

$$\begin{aligned} \dim \epsilon_\lambda \rho(Y_1 + Y_2) &= \dim \epsilon_\lambda \rho(Y_1) + \dim \epsilon_\lambda \rho(Y_2) - \dim(\epsilon_\lambda \rho(Y_1) \cap \epsilon_\lambda \rho(Y_2)) \\ &\leq \dim \epsilon_\lambda \rho(Y_1) + \dim \epsilon_\lambda \rho(Y_2) - \dim \epsilon_\lambda \rho(Y_1 \cap Y_2) \end{aligned}$$

holds and then to choose λ as a common maximal point for the submodules $\rho(Y_1 + Y_2)$, $\rho(Y_1)$, $\rho(Y_2)$ and $\rho(Y_1 \cap Y_2)$. \square

Note that, for closed invariant subspaces $Y_1, Y_2 \in \text{Lat}(T)$, the inequality in 5.1 can be rewritten as

$$\text{fd}(Y_1) + \text{fd}(Y_2) \geq \text{fd}(Y_1 \vee Y_2) + \text{fd}(Y_1 \cap Y_2).$$

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let D be an N -dimensional complex vector space. We shall say that a function $f \in \mathcal{O}(\Omega, D)$ has coefficients in a given subalgebra $A \subseteq \mathcal{O}(\Omega)$ if the coordinate functions of f with respect to some, or equivalently, every basis of D belong to A . Let $M \subseteq \mathcal{O}(\Omega, D)$ be a $\mathbb{C}[z]$ -submodule. We shall say that A is dense in M if every function $f \in M$ is the pointwise limit of a sequence $(f_k)_{k \in \mathbb{N}}$ of functions in M such that each f_k has coordinate functions in A .

Theorem 5.2. *Let $M_1, M_2 \subseteq \mathcal{O}(\Omega, D)$ be $\mathbb{C}[z]$ -submodules such that A is dense in M_1 and in M_2 and such that $AM_i \subseteq M_i$ for $i = 1, 2$. Then we have*

$$\text{fd}(M_1 + M_2) + \text{fd}(M_1 \cap M_2) = \text{fd}(M_1) + \text{fd}(M_2).$$

Proof. Exactly as in the proof of Lemma 4.1 it follows that

$$\text{fd}(M_1 + M_2) + \text{fd}(M_1 \cap M_2) \leq \text{fd}(M_1) + \text{fd}(M_2).$$

To prove the reverse inequality it suffices to check that the arguments used in [4] to prove the corresponding result for invariant subspaces of analytic functional Hilbert spaces $H(K)$ given by a complete Nevanlinna-Pick kernel on a domain in \mathbb{C} remain valid. For the convenience of the reader, we indicate the main ideas.

Define $M = M_1 + M_2$ and choose a point $\lambda \in \Omega$ which is maximal with respect to M_1 , M_2 and M . Define $E = (M_1)_\lambda \cap (M_2)_\lambda$ and choose direct complements E_1 of E in $(M_1)_\lambda$ and E_2 of E in $(M_2)_\lambda$. Fix bases (e_1, \dots, e_{d_1}) of E_1 , $(e_{d_1+1}, \dots, e_{d_1+d_2})$ for E_2 and $(e_{d_1+d_2+1}, \dots, e_{d_1+d_2+d'})$ for E , where $d_1, d_2, d' \geq 0$ are non-negative integers. Set $d = d_1 + d_2 + d'$. An elementary argument shows that (e_1, \dots, e_d) is a basis of M_λ . Let us complete this basis to a basis $B = (e_1, \dots, e_d, e_{d+1}, \dots, e_N)$ of D . Since $\text{fd}(M_1) + \text{fd}(M_2) - \text{fd}(M) = d'$, we have to show that

$$\text{fd}(M_1 \cap M_2) \geq d'.$$

We may of course assume that $d' \neq 0$. Since A is dense in M , there are functions $h_1, \dots, h_d \in M$ with

$$h_i(\lambda) = e_i \quad (i = 1, \dots, d)$$

such that each h_i has coefficients in A . Write

$$h_i = \sum_{j=1}^N h_{ij} e_j \quad (i = 1, \dots, d).$$

Then $\theta = (h_{ij})_{1 \leq i, j \leq d}$ is a $(d \times d)$ -matrix with entries in A such that $\theta(\lambda) = E_d$ is the unit matrix. By basic linear algebra there is a $(d \times d)$ -matrix (A_{ij}) with entries in A such that $(A_{ij})\theta = \text{diag}(\det \theta)$ is the $(d \times d)$ -diagonal matrix with all diagonal terms equal to $\det(\theta)$. Then

$$(A_{ij})_{1 \leq i, j \leq d} (h_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq N}} = (\text{diag}(\det \theta), (g_{ij})),$$

where (g_{ij}) is a suitable matrix with entries in A . We define functions $H_1, \dots, H_d \in M$ by setting

$$H_i = \det(\theta) e_i + \sum_{j=1}^{N-d} g_{ij} e_{d+j} = \sum_{j=1}^N \left(\sum_{\nu=1}^d A_{i\nu} h_{\nu j} \right) e_j = \sum_{\nu=1}^d A_{i\nu} h_\nu.$$

By construction $H_i(\lambda) = e_i$ and $(H_1(z), \dots, H_d(z))$ is a basis of M_z for every point $z \in \Omega$ with $\det(\theta(z)) \neq 0$. If $f = f_1 e_1 + \dots + f_N e_N \in M$ is arbitrary,

then at each point $z \in \Omega$ which is not contained in the zero set $Z(\det(\theta))$ of the analytic function $\det(\theta) \in \mathcal{O}(\Omega)$, the function f can be written as a linear combination

$$f(z) = \lambda_1(z, f)H_1(z) + \dots + \lambda_d(z, f)H_d(z).$$

Using the definition of the functions H_i , we find that

$$f_1 = \lambda_1(\cdot, f) \det(\theta), \dots, f_d = \lambda_d(\cdot, f) \det(\theta).$$

Hence, for $j = d+1, \dots, N$ and $z \in \Omega \setminus Z(\det \theta)$, we obtain that

$$\begin{aligned} f_j(z) &= \lambda_1(z, f)g_{1,j-d}(z) + \dots + \lambda_d(z, f)g_{d,j-d}(z) \\ &= \frac{g_{1,j-d}(z)}{\det \theta(z)} f_1(z) + \dots + \frac{g_{d,j-d}(z)}{\det \theta(z)} f_d(z). \end{aligned}$$

In particular, each function $f = f_1 e_1 + \dots + f_N e_N \in M$ is uniquely determined by its first d coordinate functions (f_1, \dots, f_d) .

Since A is dense in M_1 and in M_2 , we can choose functions $F_1, \dots, F_{d_1+d'} \in M_1$ and $G_1, \dots, G_{d_2+d'} \in M_2$ with coefficients in A such that

$$(F_i(\lambda))_{i=1, \dots, d_1+d'} = (e_1, \dots, e_{d_1}, e_{d_1+d_2+1}, \dots, e_{d_1+d_2+d'})$$

and

$$(G_i(\lambda))_{i=1, \dots, d_2+d'} = (e_{d_1+1}, \dots, e_{d_1+d_2+d'}).$$

Write the first d coordinate functions of each of the functions

$$F_1, \dots, F_{d_1}, G_1, \dots, G_{d_2}, F_{d_1+1}, \dots, F_{d_1+d'}, G_{d_2+1}, \dots, G_{d_2+d'}$$

with respect to the basis (e_1, \dots, e_N) of D as column vectors and arrange these column vectors to a matrix Δ in the indicated order. Then Δ is a $(d \times (d+d'))$ -matrix with entries in A . Write $\Delta = (\Delta_0, \Delta_1)$ where Δ_0 is the $(d \times d)$ -matrix consisting of the first d columns of Δ and Δ_1 is the $(d \times d')$ -matrix consisting of the last d' columns of Δ .

By construction we have $\det(\Delta_0(\lambda)) = 1$. On $\Omega \setminus Z(\det \Delta_0)$, we can write

$$(\det \Delta_0) \Delta_0^{-1} \Delta = (\text{diag}(\det \Delta_0), \Gamma),$$

where $\text{diag}(\det \Delta_0)$ is the $(d \times d)$ -diagonal matrix with all diagonal terms equal to $\det \Delta_0$ and $\Gamma = (\gamma_{ij})$ is a $(d \times d')$ -matrix with entries in A . The column vectors

$$r_j = (\gamma_{1j}, \dots, \gamma_{dj}, 0, \dots, 0, -\det \Delta_0, 0, \dots, 0)^t \quad (j = 1, \dots, d'),$$

where $-\det \Delta_0$ is the entry in the $(d+j)$ -th position, satisfy the equations

$$(\det \Delta_0) \Delta_0^{-1} \Delta r_j = ((\det \Delta_0) \gamma_{ij} - (\det \Delta_0) \gamma_{ij})_{i=1}^d = 0$$

on $\Omega \setminus Z(\det \Delta_0)$. Hence $\Delta r_j = 0$ for $j = 1, \dots, d'$, or equivalently, for each $j = 1, \dots, d$, the first d coordinate functions of

$$\gamma_{1j}F_1 + \dots + \gamma_{d_1j}F_{d_1} + \gamma_{d_1+d_2+1,j}F_{d_1+1} + \dots + \gamma_{d_1+d_2+d',j}F_{d_1+d'}$$

with respect to (e_1, \dots, e_N) coincide with those of

$$(\det \Delta_0)G_{d_2+j} - \gamma_{d_1+1,j}G_1 - \dots - \gamma_{d_1+d_2,j}G_{d_2}.$$

Since, for each j , both functions belong to M , they coincide. But then these functions belong to $M_1 \cap M_2$. Since the vectors

$$G_i(\lambda) = e_{d_1+i} \quad (i = 1, \dots, d_2 + d')$$

are linearly independent and since $\det(\Delta_0(\lambda)) = 1$, it follows that $\text{fd}(M_1 \cap M_2) = \dim(M_1 \cap M_2)_\lambda \geq d'$. \square

Suppose for the moment that $\Omega \subseteq \mathbb{C}^n$ is a Runge domain. Since by the Oka-Weil approximation theorem, the polynomials are dense in $\mathcal{O}(\Omega)$ with respect to the Fréchet space topology of uniform convergence on compact subsets, each $\mathbb{C}[z]$ -submodule $M \subseteq \mathcal{O}(\Omega, D)$ which is closed with respect to the Fréchet space topology of $\mathcal{O}(\Omega, D)$ is automatically an $\mathcal{O}(\Omega)$ -submodule. Hence we obtain the following consequence of Theorem 5.2.

Corollary 5.3. *Let $\Omega \subseteq \mathbb{C}^n$ be a Runge domain and let D be a finite-dimensional complex vector space. Then the fiber dimension formula*

$$\text{fd}(M_1 + M_2) + \text{fd}(M_1 \cap M_2) = \text{fd}(M_1) + \text{fd}(M_2)$$

holds for each pair of closed $\mathbb{C}[z]$ -submodules M_1, M_2 of the Fréchet space $\mathcal{O}(\Omega, D)$.

Suppose that $T \in L(X)^n$ is a dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^n$. Choose a CF-representation

$$\rho : X \rightarrow \mathcal{O}(\Omega_0, D)$$

of T as in the proof of Theorem 2.6. Let $M \in \text{Lat}(T)$ be an invariant subspace of T such that each vector $m \in M$ is the limit of a sequence of vectors in

$$M \cap \text{span}\{T^\alpha x; \alpha \in \mathbb{N}^n \text{ and } x \in D\}.$$

Then $\rho(M) \subseteq \mathcal{O}(\Omega_0, D)$ is a $\mathbb{C}[z]$ -submodule in which the polynomials are dense in the sense explained in the section leading to Theorem 5.2. Hence, for any two invariant subspaces $M_1, M_2 \in \text{Lat}(T)$ of this type, the fiber dimension formula

$$\begin{aligned} \text{fd}(M_1 + M_2) + \text{fd}(M_1 \cap M_2) &= \text{fd}(\rho(M_1) + \rho(M_2)) + \text{fd}(\rho(M_1) \cap \rho(M_2)) \\ &= \text{fd}(\rho(M_1)) + \text{fd}(\rho(M_2)) = \text{fd}(M_1) + \text{fd}(M_2) \end{aligned}$$

holds. The above density condition on M is trivially fulfilled for every closed T -invariant subspace M which is generated by a subset of D . But there are other situations to which this observation applies.

Recall that a commuting tuple $T \in L(H)^n$ of bounded operators on a complex Hilbert space H is called graded if $H = \bigoplus_{k=0}^{\infty} H_k$ is the orthogonal sum of closed subspaces $H_k \subseteq H$ such that $\dim H_0 < \infty$ and

- (i) $T_j H_k \subseteq H_{k+1} \quad (k \geq 0, j = 1, \dots, n),$
- (ii) $\sum_{j=1}^n T_j H \subseteq H$ is closed,
- (iii) $\bigvee_{\alpha \in \mathbb{N}^n} T^\alpha H_0 = H.$

It is elementary to show (Lemma 2.4 in [10]) that under these hypotheses the identities

$$\sum_{|\alpha|=k} T^\alpha H = \bigoplus_{j=k}^{\infty} H_j \text{ and } \sum_{|\alpha|=k} T^\alpha H_0 = H_k$$

hold for all integers $k \geq 0$. By definition a closed invariant subspace $M \in \text{Lat}(T)$ of a graded tuple $T \in L(H)^n$ is said to be homogeneous if

$$M = \bigoplus_{k=0}^{\infty} M \cap H_k.$$

Corollary 5.4. *Let $T \in L(H)^n$ be a graded dual Cowen-Douglas tuple on a domain $\Omega \subseteq \mathbb{C}^n$. Then the fiber dimension formula*

$$\text{fd}(M_1 + M_2) + \text{fd}(M_1 \cap M_2) = \text{fd}(M_1) + \text{fd}(M_2)$$

holds for any pair of homogeneous invariant subspaces $M_1, M_2 \in \text{Lat}(T)$.

Proof. By the remarks preceding the corollary

$$H = \left(\sum_{j=1}^n T_j H \right) \oplus H_0.$$

Hence in the proof of Theorem 2.6 we can choose $D = H_0$. Let $\rho : H \rightarrow \mathcal{O}(\Omega_0, H_0)$ be a CF-representation of T as constructed in the proof of Theorem 2.6. Let $M \in \text{Lat}(T)$ be a homogeneous invariant subspace for T . Then each element $m \in M$ can be written as a sum $m = \sum_{k=0}^{\infty} m_k$ with

$$m_k \in M \cap \sum_{|\alpha|=k} T^\alpha H_0 \quad (k \in \mathbb{N}).$$

Hence the assertion follows from the remarks preceding Corollary 5.4. \square

Typical examples of graded dual Cowen-Douglas tuples are multiplication tuples $M_z = (M_{z_1}, \dots, M_{z_n}) \in L(H)^n$ with the coordinate functions on analytic functional Hilbert spaces $H = H(K_f, \mathbb{C}^N)$ given by a reproducing kernel

$$K_f : B_r(a) \times B_r(a) \rightarrow L(\mathbb{C}^n), K_f(z, w) = f(\langle z, w \rangle) 1_{\mathbb{C}^N},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a one-variable power series with radius of convergence $R = r^2 > 0$ such that $a_0 = 1$, $a_n > 0$ for all n and

$$0 < \inf_{n \in \mathbb{N}} \frac{a_n}{a_{n+1}} \leq \sup_{n \in \mathbb{N}} \frac{a_n}{a_{n+1}} < \infty$$

(see [12] or [17]). In this case H is the orthogonal sum

$$H = \bigoplus_{k=0}^{\infty} \mathbb{H}_k \otimes \mathbb{C}^N$$

of the subspaces consisting of all homogeneous \mathbb{C}^N -valued polynomials of degree k and every invariant subspace

$$M = \bigvee_{i=1}^r \mathbb{C}[z] p_i \in \text{Lat}(M_z)$$

generated by a finite set of homogeneous polynomials $p_i \in \mathbb{H}_{k_i} \otimes \mathbb{C}^N$ is homogeneous. This class of examples contains the Drury-Arveson space, the Hardy space and the weighted Bergman spaces on the unit ball.

Let $H = H(K) \subseteq \mathcal{O}(\Omega)$ be an analytic functional Hilbert space on a domain $\Omega \subseteq \mathbb{C}^n$, or equivalently, a functional Hilbert space given by an analytic reproducing kernel $K : \Omega \times \Omega \rightarrow \mathbb{C}$. Let D be a finite-dimensional complex Hilbert space. Then the D -valued functional Hilbert space $H(K_D) \subseteq \mathcal{O}(\Omega, D)$ given by the kernel

$$K_D : \Omega \times \Omega \rightarrow L(D), K_D(z, w) = K(z, w) 1_D$$

can be identified with the Hilbert space tensor product $H(K) \otimes D$. Let us denote by $M(H) = \{\varphi : \Omega \rightarrow \mathbb{C}; \varphi H \subseteq H\}$ the multiplier algebra of H .

Corollary 5.5. *Suppose that $H = H(K)$ contains all constant functions and that $z_1, \dots, z_n \in M(H)$.*

(a) *For any pair of closed subspaces $M_1, M_2 \subseteq H(K_D)$ such that $M(H)M_i \subseteq M_i$ for $i = 1, 2$ and such that $M(H)$ is dense in M_1 and M_2 , the fiber dimension formula*

$$\text{fd}(M_1 \vee M_2) + \text{fd}(M_1 \cap M_2) = \text{fd}(M_1) + \text{fd}(M_2)$$

holds.

- (b) If in addition K is a complete Nevanlinna-Pick kernel, that is, K has no zeros and also the mapping $1 - \frac{1}{K}$ is positive definite, then the fiber dimension formula holds for all closed subspaces $M_1, M_2 \subseteq H(K_D)$ which are invariant for $M(H)$.

Proof. Part (a) is a direct consequence of Theorem 5.2. If K is a complete Nevanlinna-Pick kernel, then the Beurling-Lax-Halmos theorem for Nevanlinna-Pick spaces proved by McCullough and Trent (see Theorem 8.67 in [1] or Theorem 3.3.8 in [2]) implies that $M(H)$ is dense in every closed subspace $M \subseteq H(K_D)$ which is invariant for $M(H)$. □

Note that the condition that $M(H)$ is dense in a subspace $M \subseteq H(K_D)$ is satisfied for every closed $M(H)$ -invariant subspace $M \subseteq H(K_D)$ that is generated by an arbitrary family of functions $f_i : \Omega \rightarrow D$ ($i \in I$) with coefficients in $M(H)$. Part (b) for domains $\Omega \subseteq \mathbb{C}$ was proved in [3]. The proof in the multivariable case is the same.

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